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Dynamic Condensation Method for Structural Eigenvalue Analysis

Luis E. Suarez*
University of Puerto Rico, Mayaguez, Puerto Rico 00709
and
Mahendra P. Singh†
Virginia Polytechnic Institute and State University, Blacksburg, Virginia 24061

An improved dynamic condensation approach is presented for accurate calculation of structural eigenproperties. The approach is iterative. The kept and reduced degrees of freedom in the approach are related through a condensation matrix that is used to form a condensed eigenvalue problem. An initial condensation matrix can be defined in terms of the system submatrices. The eigenproperties calculated in an iterative step are utilized to update the condensation matrix. Matrix inversions or Gaussian eliminations are avoided in the updating process by employing the orthogonality of the eigenvectors. The process of updating the condensation matrix and the eigenvalue problem is repeated until a desired convergence in the eigenvalues is achieved; usually a few iterations are quite adequate. Iterations for both the lower as well as the higher modes can be performed as the condensation matrices for the two sets of modes are very simply related. Numerical examples are presented to show the applicability of the proposed approach. Several methods that have been used in practice to select the kept degrees of freedom for condensing are also evaluated numerically with respect to their effectiveness in providing accurate estimates of the eigenproperties with a minimum number of iterations.

Introduction

IN the calculation of a dynamic response of a linear structure by the finite element method, the major computational effort is devoted to the calculation of natural frequencies and modes of vibration. For a large structure, the complete solution of its algebraic eigenvalue problem to obtain its frequencies and modes can be computationally expensive. This has motivated the development of techniques to calculate only a first few eigenproperties that are also usually the most important quantities required for a dynamic analysis. The techniques that are used for this purpose can be broadly classified into three categories: 1) substructure synthesis or modal synthesis methods, 2) Ritz analysis and vector iteration methods, and 3) dynamic reduction or dynamic condensation methods.

This paper is concerned with a condensation approach. The dynamic condensation or dynamic reduction methods (sometimes also referred to as eigenvalue economizer or mass condensation techniques) aim at reducing the size of the eigenvalue problem before proceeding with its solution. Probably the first and most widely used dynamic reduction techniques were introduced by Guyan[1] and Irons.[2] Guyan's idea has prompted several proposals for further improvements in the dynamic reduction techniques.[3-7] These methods provide approximate eigenproperties. More refined condensation methods that can provide the exact eigenvalues and eigenvectors are also available.[8-11] In these methods, the reduced system matrices are frequency dependent. The condensed eigenproblems are, thus, nonlinear, which requires special techniques for their solutions. Some iterative methods that can provide accurate estimates of the eigenproperties have also been proposed.[12-15] A comprehensive review of the material published on the subject of dynamic condensation before the early 1980s can be found in Ref. 14.

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*Assistant Professor, Department of General Engineering.
†Professor, Department of Engineering Science and Mechanics.

In this paper, an improved dynamic condensation method is presented. The method is iterative. It starts with a trial condensation matrix to form a condensed eigenvalue problem. The eigensolution obtained in an iteration step is used to improve the condensation matrix by taking advantage of a particular form of the orthogonality conditions of the eigenvectors. No Gaussian elimination or matrix inversions are required to upgrade the condensation matrix. Unlike other dynamic condensation techniques, the proposed iteration process can be carried out not only with the lower set of modes but also with the higher set of modes and, if desired, even with both sets alternatively.

In a condensation method, the selection of the kept and reduced degrees of freedom is often quite crucial for the success of the method. Several schemes for the selection of these degrees of freedom have been proposed in the literature on this subject. Although the process of selection is not as crucial for the success of the method proposed herein as it is for other classical condensation methods, it can still affect the number of iterations required to achieve the convergence to the correct results. Herein, therefore, the use of some commonly used selection schemes with the proposed approach is examined through several numerical examples.

Condensed Eigenvalue Problems

The algebraic eigenvalue problem associated with the equations of motion of the finite element model of a structure

$$[\hat{K}][\hat{\Phi}] = [\hat{M}][\hat{\Phi}][\Lambda] \tag{1}$$

can always be tranformed into the following standard eigenvalue problem

$$[K][\Phi] = [\Phi][\Lambda] \tag{2}$$

by using the transformation

$$[\hat{\Phi}] = [S]^{-1}[\Phi] \tag{3}$$

which employs the matrix $[S]$ obtained in the Cholesky decomposition[16] of matrix $[\hat{M}]$ as follows,

$$[\hat{M}] = [S]^T[S] \tag{4}$$

The matrix $[K]$ in Eq. (2) is then defined as

$$[K] = [S]^{-T}[\hat{K}][S]^{-1} \tag{5}$$

Often in finite element modeling of a structural system, several degrees of freedom are associated with negligible or zero inertial forces. Here it is assumed that such degrees of freedom have been eliminated by a static condensation procedure to render a positive definite mass matrix $[\hat{M}]$ for Eq. (1).

Let N be the total number of dynamic degrees of freedom of the system and, therefore, the size of eigenproblem (2). It is proposed to reduce the size of this eigenvalue problem to n_k by condensing (or reducing) the remaining $n_r = N - n_k$ degrees of freedom. Subscripts k and r, henceforth, refer to the kept and reduced degrees of freedom, respectively. According to this division into the kept and reduced degrees of freedom, we also subdivide the matrices in Eq. (2) as,

$$\begin{bmatrix} K_{kk} & K_{kr} \\ K_{rk} & K_{rr} \end{bmatrix}\begin{bmatrix} \Phi_{kk} & \Phi_{kr} \\ \Phi_{rk} & \Phi_{rr} \end{bmatrix} = \begin{bmatrix} \Phi_{kk} & \Phi_{kr} \\ \Phi_{rk} & \Phi_{rr} \end{bmatrix}\begin{bmatrix} \Lambda_k & 0 \\ 0 & \Lambda_r \end{bmatrix} \tag{6}$$

where the submatrices $[\Phi_{kk}]$ and $[K_{kk}]$ are square matrices of dimensions $(n_k \times n_k)$. Similarly, $[\Phi_{rr}]$ and $[K_{rr}]$ are also square matrices of size $(n_r \times n_r)$. The remaining submatrices $[\Phi_{rk}]$, $[\Phi_{kr}]$, $[K_{rk}]$, and $[K_{kr}]$ are of dimensions either $(n_r \times n_k)$ or $(n_k \times n_r)$. As this choice of division into kept and reduced degrees of freedom has a direct influence on the accuracy and rate of convergence of the numerical procedure to be presented, this will be addressed in more detail in a later section.

Equation (6) is equivalent to the following four matrix equations:

$$[K_{kk}\Phi_{kk} + K_{kr}\Phi_{rk}] = [\Phi_{kk}][\Lambda_k] \tag{7}$$

$$[K_{kk}\Phi_{kr} + K_{kr}\Phi_{rr}] = [\Phi_{kr}][\Lambda_r] \tag{8}$$

$$[K_{rk}\Phi_{kk} + K_{rr}\Phi_{rk}] = [\Phi_{rk}][\Lambda_k] \tag{9}$$

$$[K_{rk}\Phi_{kr} + K_{rr}\Phi_{rr}] = [\Phi_{rr}][\Lambda_r] \tag{10}$$

We normalize the eigenvectors in matrix $[\Phi]$ such that they have unitary euclidean norm. That is,

$$[\Phi]^T[\Phi] = \begin{bmatrix} \Phi_{kk}^T & \Phi_{rk}^T \\ \Phi_{kr}^T & \Phi_{rr}^T \end{bmatrix}\begin{bmatrix} \Phi_{kk} & \Phi_{kr} \\ \Phi_{rk} & \Phi_{rr} \end{bmatrix} = \begin{bmatrix} I_k & 0 \\ 0 & I_r \end{bmatrix} \tag{11}$$

where $[I_k]$ and $[I_r]$ are the unit matrices of order $(n_k \times n_k)$ and $(n_r \times n_r)$, respectively. Equation (11) is equivalent to four normality conditions that relate the submatrices $[\Phi_{kk}]$, $[\Phi_{kr}]$, etc., with each other. One of these equations, associated with one of the off-diagonal null matrices on the right side of Eq. (11), can be written as

$$[\Phi_{kk}^T\Phi_{kr} + \Phi_{rk}^T\Phi_{rr}] = [0] \tag{12}$$

The matrices $[\Phi_{kk}]$ and $[\Phi_{rr}]$ can now be related to $[\Phi_{rk}]$ and $[\Phi_{kr}]$ through two condensation matrices as follows,

$$[\Phi_{rk}] = [R][\Phi_{kk}] \tag{13}$$

$$[\Phi_{kr}] = [\hat{R}][\Phi_{rr}] \tag{14}$$

where the condensation matrices $[R]$ and $[\hat{R}]$ are defined as,

$$[R] = [\Phi_{rk}][\Phi_{kk}^{-1}], \quad [\hat{R}] = [\Phi_{kr}][\Phi_{rr}^{-1}] \tag{15}$$

Substituting Eqs. (13) and (14) in Eq. (12), one obtains the relationship between $[R]$ and $[\hat{R}]$ as

$$[\hat{R}] = -[R^T] \tag{16}$$

Now with this, Eq. (14) can also be written in terms of $[R]$ as

$$[\Phi_{kr}] = -[R^T][\Phi_{rr}] \tag{17}$$

From Eq. (6), the eigenvalue problem corresponding to the kept n_k modes can be written as

$$\begin{bmatrix} K_{kk} & K_{kr} \\ K_{rk} & K_{rr} \end{bmatrix}\begin{bmatrix} \Phi_{kk} \\ \Phi_{rk} \end{bmatrix} = \begin{bmatrix} \Phi_{kk} \\ \Phi_{rk} \end{bmatrix}[\Lambda_k] \tag{18}$$

whereas the eigenproblem corresponding to the n_r remaining modes is written as

$$\begin{bmatrix} K_{kk} & K_{kr} \\ K_{rk} & K_{rr} \end{bmatrix}\begin{bmatrix} \Phi_{kr} \\ \Phi_{rr} \end{bmatrix} = \begin{bmatrix} \Phi_{kr} \\ \Phi_{rr} \end{bmatrix}[\Lambda_r] \tag{19}$$

If the condensation matrix $[R]$ were known, both eigenvalue problems (18) and (19) can be reduced to their condensed forms. To do this, one defines the kept modes in terms of $[\Phi_{kk}]$ as

$$\begin{bmatrix} \Phi_{kk} \\ \Phi_{rk} \end{bmatrix} = \begin{bmatrix} I_k \\ R \end{bmatrix}[\Phi_{kk}] \tag{20}$$

Using Eq. (20) in Eq. (18) and premultiplying by $[I_k^T|R^T]$ one obtains

$$[K^*][\Phi_{kk}] = [M^*][\Phi_{kk}][\Lambda_k] \tag{21}$$

where the condensed stiffness and mass matrices $[K^*]$ and $[M^*]$ are, respectively,

$$[K^*] = [K_{kk} + R^TK_{rk} + (R^TK_{rk})^T + R^TK_{rr}R] \tag{22a}$$

$$[M^*] = [I_k + R^TR] \tag{22b}$$

Similarly, to condense Eq. (19), we define the reduced set of modes in terms of $[\Phi_{rr}]$ as,

$$\begin{bmatrix} \Phi_{kr} \\ \Phi_{rr} \end{bmatrix} = \begin{bmatrix} -R^T \\ I_r \end{bmatrix}[\Phi_{rr}] \tag{23}$$

Replacing Eq. (23) in Eq. (19) and multiplying on the left by $[-R|I_r^T]$, the following condensed eigenproblem is obtained:

$$[K'][\Phi_{rr}] = [M'][\Phi_{rr}][\Lambda_r] \tag{24}$$

where

$$[K'] = [K_{rr} - K_{rk}R^T - (K_{rk}R^T)^T + RK_{kk}R^T] \tag{25a}$$

$$[M'] = [I_r + RR^T] \tag{25b}$$

Equations (21) and (24) define the two condensed complementary eigenvalue problems. Once these two equations are solved for $[\Phi_{kk}]$ and $[\Phi_{rr}]$, the complete $(N \times N)$ eigenvector matrix $[\Phi]$ can be obtained from

$$[\Phi] = \begin{bmatrix} I_k & -R^T \\ R & I_r \end{bmatrix}\begin{bmatrix} \Phi_{kk} & 0 \\ 0 & \Phi_{rr} \end{bmatrix} \tag{26}$$

However, to define the matrices in the condensed eigenvalue problem, one needs to know $[R]$. The formal definition of the condensation matrix $[R]$ in Eqs. (15) cannot be used to compute it since the eigenvectors are not known a priori. But, as shown in the next section, an iterative procedure can be devised to compute the condensation matrix $[R]$. Once $[R]$ is known, the total eigenvectors and eigenvalues of the structural system can be obtained by solving Eqs. (21) and (24) and then using Eqs. (26) and (3).

Dynamic Condensation Matrix

An initial approximate value for the condensation matrix $[R]$ can be obtained from Eq. (9) by assuming that $[\Phi_{rk}][\Lambda_k] = [0]$. This gives

$$[\Phi_{rk}^{(o)}] = -[K_{rr}^{-1}K_{rk}\Phi_{kk}^{(o)}] \quad (27)$$

The superscript (o) on a quantity is used to indicate that this is an initial approximate value. Comparing Eq. (27) with Eq. (13), one obtains the initial approximation for $[R]$ as

$$[R^{(o)}] \simeq [K_{rr}^{-1}K_{rk}] \quad (28)$$

If $[R]$, as defined by Eq. (28), is utilized in Eqs. (22), the resultant eigenproblem (21) will have same form as the one obtained with Guyan's reduction method.[1] Let $[\Lambda_k^{(o)}]$ and $\Phi_{kk}^{(o)}]$ be the matrices of eigenvalues and eigenvectors, respectively, obtained as the solution of this approximate eigenvalue problem. This approximate solution can now be used to obtain an improved matrix $[R]$, as described in the following. For this, we consider Eq. (9) again, but now with the right side not equal to zero:

$$[K_{rr}\Phi_{rk}] = [\Phi_{rk}\Lambda_k - K_{rk}\Phi_{kk}] \quad (29)$$

wherein the first term on the right side of Eq. (29) is the correction term to be added to improve the original estimate.

Replacing Eq. (13) in the right side of Eq. (29) and solving for $[\Phi_{rk}]$, we obtain

$$[\Phi_{rk}] = [K_{rr}^{-1}][R\Phi_{kk}\Lambda_k - K_{rk}\Phi_{kk}] \quad (30)$$

Postmultiplying Eq. (30) by $[\Phi_{kk}^{-1}]$ and recalling the definition of $[R]$, Eqs. (15), one obtains

$$[R] = [K_{rr}^{-1}][R\Phi_{kk}\Lambda_k\Phi_{kk}^{-1} - K_{rk}] \quad (31)$$

One of the terms inside the brackets in Eq. (31) requires the inversion of $[\Phi_{kk}]$. To avoid the calculation of this inverse, we utilize the orthogonality property of the eigenvector matrix $[\Phi_{kk}]$ with respect to $[K^*]$:

$$[\Phi_{kk}^T K^*\Phi_{kk}] = [\Lambda_k] \quad (32)$$

By pre- and postmultiplying Eq. (32) by $[\Phi_{kk}]$ and $[\Phi_{kk}^{-1}]$, respectively, one obtains

$$[\Phi_{kk}\Lambda_k\Phi_{kk}^{-1}] = [\Phi_{kk}\Phi_{kk}^T K^*] \quad (33)$$

By substituting Eq. (33) in Eq. (31), one obtains

$$[R] = [K_{rr}^{-1}][R\Phi_{kk}\Phi_{kk}^T K^* - K_{rk}] \quad (34)$$

If the actual condensation matrix $[R]$ and eigenvector matrix $[\Phi_{kk}]$ are used in the right side of Eq. (34), the equality will be identically satisfied. However, one can also use Eq. (34) to define a recursive relationship to obtain improved estimates of $[R]$ iteratively as follows,

$$[R^{(l+1)}] = [K_{rr}^{-1}][R^{(l)}\Phi_{kk}^{(l)}\Phi_{kk}^{(l)^T}K^{*(l)} - K_{rk}] \quad (35)$$

In this equation, one can use the approximate matrices $[R^{(o)}]$ and $[\Phi_{kk}^{(o)}]$ and obtain a new estimate for $[R]$, which will be identified as $[R^{(1)}]$. This improved condensation matrix $[R^{(1)}]$ can then be employed in Eqs. (22) to calculate new condensed stiffness and mass matrices as

$$[K^*]^{(l)} = [K_{kk} + R^{(l)^T}K_{rk} + (R^{(l)^T}K_{rk})^T + R^{(l)^T}K_{rr}R^{(l)}] \quad (36a)$$

$$[M^*]^{(l)} = [I_k + R^{(l)^T}R^{(l)}] \quad (36b)$$

where $l = 1$ for the first iteration. A new eigenvalue problem (21) is formed with these matrices

$$[K^*]^{(l)}[\Phi_{kk}]^{(l)} = [M^*]^{(l)}[\Phi_{kk}]^{(l)}[\Lambda_k]^{(l)} \quad (37)$$

The eigenvector matrix $[\Phi_{kk}]^{(l)}$ obtained from the solution of the eigenproblem is used to calculate a new condensation matrix $[R^{(l+1)}]$ by using Eq. (35). The procedure continues by defining the new systems matrices $[K^*]$ and $[M^*]$ from Eqs. (36), solving the eigenproblem (37), and recalculating the condensation matrix $[R]$ until the desired convergence is achieved. The convergence criterion can be in terms of a tolerance limit on the eigenvalues $[\Lambda_k]$ calculated at two consecutive iteration steps.

Since the condensation matrices of the two complementary eigenvalue problems are simply related by Eq. (16), the matrix $[R]$ obtained in the final iteration step can also be used to define the condensed eigenvalue problem associated with the reduced modes, Eq. (24). Finally, the total modal matrix $[\Phi]$ of the structural system can be obtained from Eq. (26).

The foregoing procedure allows one to calculate the total eigensolution of the system by iterating with the lower modes only. However, as it will be shown next, it is also possible to obtain the same results by iterating with the higher modes, or iterating alternatively with both the lower and higher modes, if desired.

Condensation with Higher Modes

While condensing with the higher modes, the initial condensation matrix is still obtained as before by using Eq. (28). One then calculates the condensed matrices $[K']$ and $[M']$ from Eqs. (25), and solves eigenproblem (24) associated with the higher modes. This provides a first approximation for $[\Phi_{rr}]$. This result can then be used to improve the initial estimate $[R]$ by a recursive relationship as follows.

To develop a recursive relationship like Eq. (35) in terms of $[\Phi_{rr}]$ instead of $[\Phi_{kk}]$, one substitutes Eq. (17) into Eq. (8) and obtains

$$-[R^T\Phi_{rr}\Lambda_r] = [-K_{kk}R^T + K_{kr}][\Phi_{rr}] \quad (38)$$

Multiplying Eq. (38) on the right by $[\Phi_{rr}^{-1}]$,

$$[R^T\Phi_{rr}\Lambda_r\Phi_{rr}^{-1}] = [K_{kk}R^T - K_{kr}] \quad (39)$$

and solving for $[R]$ on the left side, one obtains

$$[R]^T = [K_{kk}R^T - K_{kr}][\Phi_{rr}][\Lambda_r^{-1}][\Phi_{rr}^{-1}] \quad (40)$$

The transpose of Eq. (40) provides

$$[R] = [\Phi_{rr}^{-T}\Lambda_r^{-1}\Phi_{rr}^T][RK_{kk} - K_{rk}] \quad (41)$$

Again, to avoid the calculation of the inverse of $[\Phi_{rr}]$ required in Eq. (41), one can utilize the orthonormal properties of $[\Phi_{rr}]$ as

$$[\Phi_{rr}^T M'\Phi_{rr}] = [I_r] \quad (42)$$

from which one can express the inverse of $[\Phi_{rr}]$ as,

$$[\Phi_{rr}^{-T}] = [M'\Phi_{rr}] \quad (43)$$

Using Eq. (43) in Eq. (41), one obtains

$$[R] = [M'\Phi_{rr}\Lambda_r^{-1}\Phi_{rr}^T][RK_{kk} - K_{rk}] \quad (44)$$

This equation will be identically satisfied if the actual eigenvalues $[\Lambda_r]$, eigenvector matrix $[\Phi_{rr}]$, and condensation matrix $[R]$ are used. One can also use Eq. (44) to define a recursive relationship to obtain an improved estimate of $[R]$ iteratively as follows,

$$[R^{(l+1)}] = [M'^{(l)}][\Phi_{rr}^{(l)}][\Lambda_r^{-1}]^{(l)}[\Phi_{rr}^{(l)^T}][R^{(l)}K_{kk} - K_{rk}] \quad (45)$$

One will use the approximate values of $[\Lambda_r]$, $[\Phi_{rr}]$, and $[R]$, calculated at a given step, in the right side of Eq. (45) to obtain a better estimate of $[R]$. At any iterative step, the new matrix $[R]$ is, in turn, used to recalculate the system matrices $[K']$ and $[M']$ corresponding to the eigenproblem associated with the higher modes as follows,

$$[K']^{(l)} = [K_{rr} - K_{rk}R^{(l)^T} - (K_{rk}R^{(l)^T})^T + R^{(l)}K_{kk}R^{(l)^T}] \quad (46a)$$

$$[M']^{(l)} = [I_r + R^{(l)}R^{(l)^T}] \quad (46b)$$

These matrices are then used in the eigenvalue problem:

$$[K']^{(l)}[\Phi_{rr}]^{(l)} = [M']^{(l)}[\Phi_{rr}]^{(l)}[\Lambda_r]^{(l)} \quad (47)$$

the solution of which provides the eigenvalues $[\Lambda_r^{(l)}]$ and eigenvector matrix $[\Phi_{rr}^{(l)}]$. These are used again to improve the estimate of $[R]$ according to Eq. (45). Once a predecided convergence criterion on the calculated eigenvalues has been satisfied, the remaining elements of the n_r higher eigenvectors are calculated from Eq. (17). The matrix $[R]$, obtained in the final iteration step, is then used to define the condensed matrices for the lower modes using Eqs. (22). The solution of the condensed eigenvalue problem, Eq. (21), then provides the eigenvectors associated with the lower modes. Finally, Eqs. (26) and (3) can be used to obtain the complete set of modes.

The difference in the form of Eqs. (35) and (45), which are used to define the updated condensation matrices, may be noted. One could also choose to define $[R]$ from Eq. (39) as

$$[R^T] = [K_{kk}^{-1}][K_{kr} + R^T\Phi_{rr}\Lambda_r\Phi_{rr}^{-1}] \quad (48)$$

instead of Eq. (40) and proceed further to establish an interative algorithm, similar to Eq. (35) to update $[R]$. However, it so happens that this formulation will not lead to convergence because the corrective term defined by the second term on the right side of Eq. (48) is of significantly larger magnitude as it involves the higher eigenvalues of the system. This was verified numerically.

Selection of Degrees of Freedom for Dynamic Condensation

An important decision one has to make in the implementation of a condensation procedure is about the selection of the kept and reduced degrees of freedom. In some condensation procedures, if care is not exercised in the selection of the kept degrees of freedom, some of the lower eigenvalues may even be lost from the reduced spectrum.[17] In the proposed approach, it is also necessary to identify the kept degrees of freedom to define the condensation matrix $[R]$ from Eq. (28). It is noted that Eq. (28) is only one possible way to define the initial matrix for the iterative process. There may be other, possibly more efficient, methods to define $[R^{(o)}]$. Although the proposed iterative process will converge to the set of eigenvalues sought independently of the initial choice of the matrix $[R^{(o)}]$, the choice of the kept degrees of freedom can still affect the convergence rate significantly.

Some guidelines are available in the literature for the selection of the master degrees of freedom in Guyan's technique.[3,17–19] Most of these guidelines are of qualitative nature. For example, suggestions have been offered to select the master degrees of freedom as 1) the degrees of freedom with the largest entries in the mass matrix, 2) displacement degrees of freedom rather than rotations, 3) the degrees of freedom that conserve the greatest possible strain energy, etc. Another very commonly used criterion is to select the degrees of freedom with the lowest stiffness to mass ratios, that is, $r_i = K_{ii}/M_{ii}$ in the system matrices. In situations where there are several degrees of freedom with equal ratios, those with larger M_{ii} terms are preferred. Also, only a limited number of these degrees of freedom are used if they are located in the same region of the structure.

Two more specific and rational techniques for selecting the master degrees of freedom have been proposed by Shah and Raymund[20] and Matta.[21] Their algorithms can be programmed on a computer for an automatic selection of the degrees of freedom. Both of these techniques use the ratio r_i mentioned earlier. In Shah and Raymund's procedure, the degrees of freedom with the largest ratio are eliminated by applying the Guyan reduction scheme, and new reduced stiffness and mass matrices are obtained. In Matta's procedure, the degrees of freedom with the largest ratios are also eliminated. To account for the effect of each eliminated degree of freedom, say the pth degree of freedom, on the remaining degrees of freedom, the diagonal terms of the stiffness and mass matrices are modified as follows,

$$K_{ii} = K_{ii} - (K_{ip})^2/K_{pp}$$

$$M_{ii} = M_{ii} + (K_{ip}/K_{pp})^2M_{pp}$$

In the numerical example considered here, the three techniques described earlier have been used and compared with regard to their effect on the rate of convergence of the eigensolution.

Numerical Results

To demonstrate the application of the proposed dynamic condensation procedure, the first example problem considered

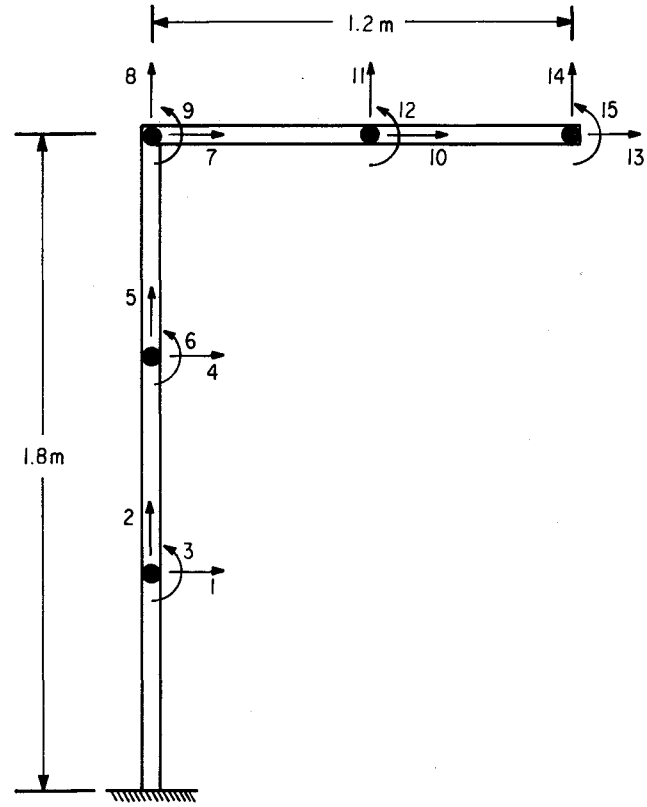

Fig. 1 Schematic of a 15-degree-of-freedom frame.

Table 1 Natural frequencies of the two-dimensional frame in Fig. 1

Frequency number	Natural frequency, rad/s
1	0.29888
2	0.97399
3	3.78910
4	7.47794
5	11.29715
6	22.60495
7	29.17504
8	46.61875
9	66.21488
10	98.52548
11	430.58509
12	1287.81227
13	1435.52017
14	2447.25333
15	2798.91326

Table 2 Frequencies of frame in Fig. 1 calculated by iterating with lower modes and with kept degrees of freedom selected according to the ratio r_i

Frequency number	Iteration number[a]					
	0	1	2	3	4	5
1	0.29907	0.29888	0.29888	0.29888	0.29888	0.29888
	(0.064)	(0.000)	—	—	—	—
2	0.97540	0.97399	0.97399	0.97399	0.97399	0.97399
	(0.145)	(0.000)	—	—	—	—
3	3.87125	3.78910	3.78910	3.78910	3.78910	3.78910
	(2.168)	(0.000)	—	—	—	—
4	7.51341	7.47797	7.47794	7.47794	7.47794	7.47794
	(0.474)	(0.000)	—	—	—	—
5	12.08771	11.29995	11.29746	11.29720	11.29716	11.29715
	(6.998)	(0.025)	(0.003)	(0.000)	—	—
6	23.60249	22.66340	22.61416	22.60701	22.60549	22.60509
	(4.413)	(0.259)	(0.041)	(0.009)	(0.002)	(0.001)
7	32.90413	29.46312	29.26032	29.20414	29.18556	29.17896
	(12.782)	(0.987)	(0.292)	(0.100)	(0.036)	(0.013)

[a]Percent errors in the frequencies shown in parentheses.

Table 3 Percent errors in the frequencies of frame in Fig. 1 calculated by iterating with higher modes and with kept degrees of freedom selected according to the ratio r_i

Frequency number	Iteration number					
	0	1	2	3	4	5
8	−3.076	−0.613	−0.220	−0.135	−0.113	−0.106
9	−1.531	−0.245	−0.076	−0.048	−0.042	−0.040
10	−0.166	−0.013	−0.004	−0.003	−0.003	−0.003

Table 4 Comparison of percent errors in the frequencies obtained with four different selection schemes for the kept degrees of freedom

Iteration step	Scheme number	Frequency number						
		1	2	3	4	5	6	7
0	1	——	——	0.083	0.226	0.252	7.490	11.104
	2	0.020	0.049	0.630	0.068	2.522	4.618	27.866
	3	0.070	0.143	2.825	0.909	8.641	36.200	56.375
	4	0.619	31.1	40.3	88.0	200.1	1657	4886
1	1	——	——	——	——	0.001	0.001	1.877
	2	——	——	——	——	0.003	0.608	4.581
	3	——	——	——	0.001	0.064	2.577	1.266
	4	——	——	0.017	0.930	1.191	13.251	1575
2	1	——	——	——	——	——	——	0.468
	2	——	——	——	——	——	0.140	1.512
	3	——	——	——	——	0.015	1.408	0.383
	4	——	——	0.006	0.015	0.233	5.519	7.319
5	1	——	——	——	——	——	——	0.015
	2	——	——	——	——	——	0.008	0.154
	3	——	——	——	——	0.001	0.149	0.032
	4	——	——	0.002	0.013	0.088	2.554	2.173

here is the frame structure shown in Fig. 1. The structure has a total of 15 degrees of freedom with 3 degrees of freedom per node. The properties of the frame are the following: cross sectional area = 16 cm^2, area moment of inertia = 21.333 cm^4, modulus of elasticity = 210 GPa, and mass density = 0.00783 kg/cm^3. The natural frequencies of this structure calculated without any condensation are given in Table 1. They are considered as "exact" values for comparison purposes.

Out of a total of 15, 7 degrees of freedom corresponding to the smallest value for the ratio $r_i = K_{ii}/M_{ii}$ in Eq. (1) are chosen as the kept degrees of freedom for condensing Eq. (2). These degrees of freedom are 1, 3, 4, 9, 11, 12, and 14, and they are shown in Fig. 1. The natural frequencies calculated at each step of iteration with the lower modes are listed in Table 2. The percent errors in the frequencies defined as $100 \times (\omega_{appr} - \omega_{exact})/\omega_{exact}$ are shown in parentheses in the table. It is seen that one iteration is enough to obtain all of the frequencies with errors less than 1%. A frequency with an error less than 0.0005% is indicated by dashes in this and other tables. After three iterations, five frequencies have errors below 0.0005%. All of the error values are positive, indicating that the calculated frequencies are higher than the exact values. That is, the eigenvalues converged from above in this case.

The errors in the frequencies obtained by iterating with the higher modes are presented in Table 3. Frequencies 11–15 all had errors less than 0.0005 even in the first iteration; therefore, they are not shown in the table to save some space. Here again, after the first iteration, the complete frequency spectrum has absolute errors less than 1%. In this case, all of the error values are negative, indicating that when the condensation process is carried out with the higher modes the calculated frequencies are lower than the exact frequencies. In other words, the eigenvalues in this case converge from below.

It is also of interest to examine the accuracy of the eigenvectors calculated in the proposed condensation approach. For this, the rms errors in the eigenvectors, defined as follows, were calculated:

$$e_j = 100\|\phi_{ja} - \phi_{je}\|/\|\phi_{je}\| \tag{49}$$

In Eq. (49), ϕ_{je} and ϕ_{ja}, respectively, are the exact and ap-

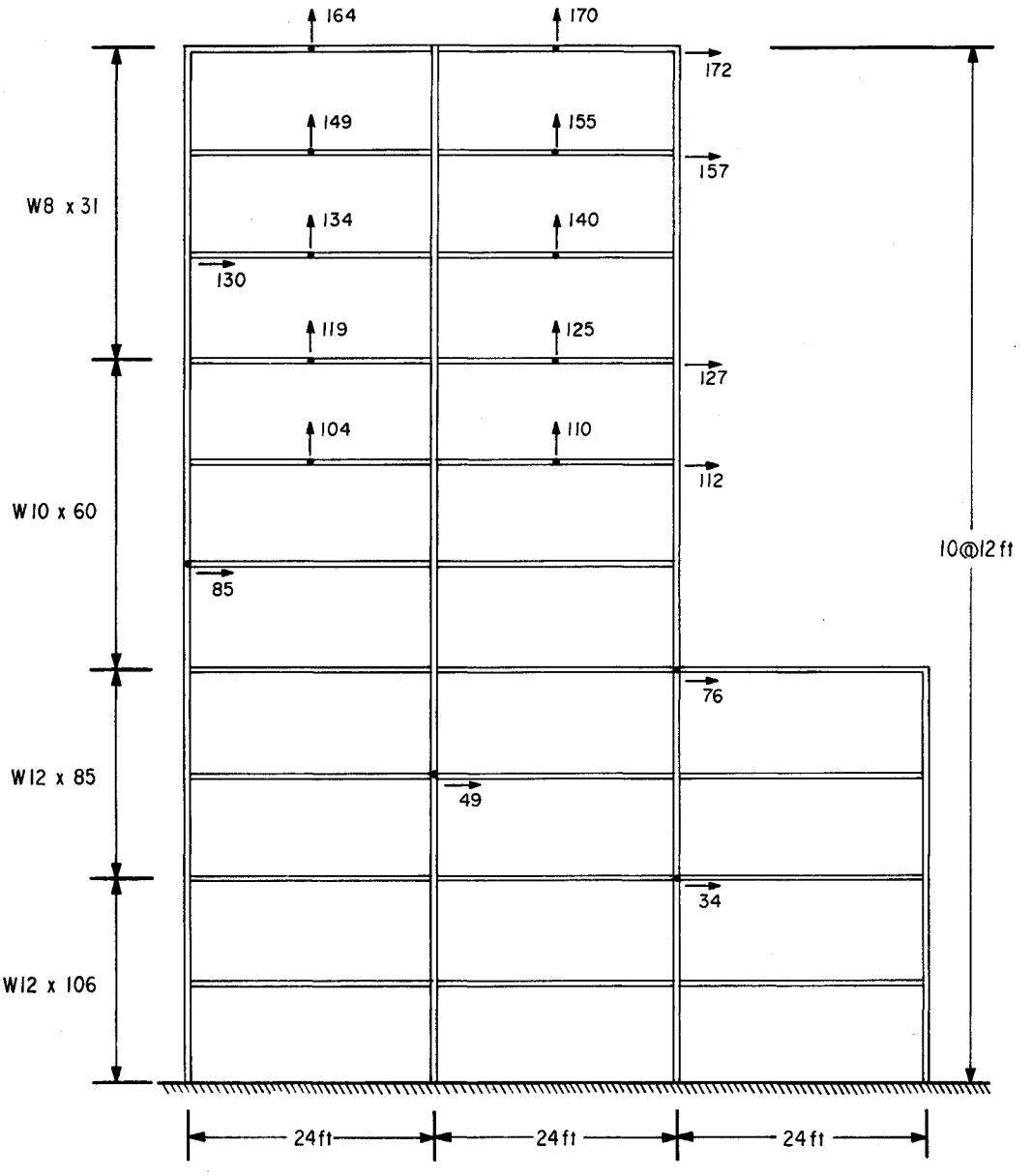

Fig. 2 Schematic of a 10-story, 174-degree-of-freedom frame.

Table 5 First 10 lower natural frequencies of the 174 DOF frame in Fig. 2

Frequency number	Natural frequency, rad/s
1	28.53695
2	65.13999
3	108.56103
4	156.51218
5	205.94206
6	238.90581
7	280.93398
8	303.29122
9	365.78086
10	376.14352

proximate eigenvectors for the jth mode and the vector norm represents the Euclidian length. For the example problem considered earlier, the rms errors in the first seven eigenvectors calculated in the last iteration of the condensation with the lower modes were, respectively, (<0.0005), (<0.0005), (0.001), (0.004), (0.060), (0.944), and (4.260). These percent errors in the eigenvectors are larger than the errors in the corresponding frequencies listed in the last column in Table 2. Similar results were also observed in the condensation with the higher modes. The rms error in the eight higher eigenvectors obtained in the last iteration were, respectively, (4.605), (3.056), (0.554), (0.013), (0.001), (0.002), (<0.0005), and (<0.0005). These errors are also higher than the errors in the last column of Table 3 for the corresponding frequencies. These observations are akin to the well-known results that, according to the Rayleigh quotient, a first-order error in an eigenvector translates into a second-order error in the corresponding eigenvalue. In the proposed method, the condensed eigenvalue problem at each iteration step is defined in terms of the condensation matrix $[R]$, which in turn is defined in terms of the approximate eigenvectors obtained in the previous step. These results indicate that an error in $[R]$ causes only a smaller error in the eigenvalues. Or conversely stated, this implies that smaller errors in the eigenvalues will be associated with larger errors in the eigenvectors.

With the example problem just mentioned, four other coordinate selections for the kept degrees of freedom have been considered. The first set, consisting of the degrees of freedom 1, 4, 6, 7, 11, 12, and 14, was selected according to Shah and Raymund's procedure.[20] The second set, consisting of 1, 4, 6, 9, 11, 12, and 14, was selected according to Matta's

Table 6 Percent errors in the first 10 frequencies of the 174 DOF frame in Fig. 2 obtained with the kept degrees of freedom selected according to the ratio r_i or Matta's method

Frequency number	Iteration number					
	0	1	3	5	7	9
1	453.6	0.024	0.019	0.016	0.015	0.013
2	458.1	3.243	1.580	1.221	0.997	0.840
3	259.0	22.779	4.799	2.870	2.013	1.529
4	159.1	60.165	5.934	3.215	2.167	1.607
5	105.6	52.394	15.367	6.085	3.797	2.780
6	90.97	49.526	18.999	5.904	3.420	2.397
7	73.14	43.277	11.008	10.705	8.681	5.022
8	85.051	38.590	26.159	13.531	2.464	2.212
9	65.861	52.092	26.486	15.795	8.294	6.273
10	69.020	55.921	30.335	17.798	14.871	12.691

Table 7 Percent errors in the first 10 frequencies of the 174 DOF frame in Fig. 2 obtained with the kept degrees of freedom selected with Shah and Raymund's scheme

Frequency number	Iteration number					
	0	1	3	5	7	9
1	0.033	0.000	—	—	—	—
2	0.177	0.000	—	—	—	—
3	0.564	0.000	—	—	—	—
4	1.071	0.001	0.000	—	—	—
5	2.905	0.031	0.000	—	—	—
6	3.143	0.037	0.001	0.000	—	—
7	5.436	0.177	0.008	0.001	0.000	—
8	30.848	4.592	4.592	1.904	0.946	0.528
9	13.137	3.844	2.931	3.388	2.778	2.763
10	35.572	18.324	15.269	14.508	13.875	12.787

Table 8 Percent errors in the lower frequencies of the frame in Fig. 2 calculated with condensed eigenvalue problems of increasing size

Number of kept modes	Iteration number					
	1	2	3	4	5	6
	First natural frequency: 28.53695 rad/s					
5	0.387	—	—	—	—	—
10	0.033	—	—	—	—	—
15	0.032	—	—	—	—	—
20	0.032	—	—	—	—	—
	Third natural frequency: 108.56103 rad/s					
5	3.245	0.040	0.003	—	—	—
10	0.564	—	—	—	—	—
15	0.454	—	—	—	—	—
20	0.451	—	—	—	—	—
	Fifth natural frequency: 205.94206 rad/s					
5	13.827	4.110	2.218	1.308	0.808	0.514
10	2.905	0.031	0.003	—	—	—
15	1.635	0.001	—	—	—	—
20	1.632	0.001	—	—	—	—
	Seventh natural frequency: 280.93398 rad/s					
10	5.436	0.177	0.030	0.008	0.003	0.001
15	3.125	0.005	—	—	—	—
20	3.100	0.004	—	—	—	—
	Ninth natural frequency: 365.78086 rad/s					
10	13.137	3.844	3.152	2.931	2.847	2.809
15	7.128	2.130	0.782	0.316	0.138	0.063
20	4.546	0.725	0.166	0.045	0.013	0.004

procedure.[21] In the third set, the degrees of freedom with the smallest elements in the main diagonal of the transformed matrix $[K]$ in Eq. (2) were elected as the kept degrees of freedom; they were 1, 3, 4, 6, 11, 14, and 15. For the last set, on the other hand, no specific selection procedure was used; rather, the first seven degrees of freedom in the node numbering scheme used in the structural model were chosen as the kept degrees of freedom. They happen to be the three degrees of freedom at the two nodes of the column and the horizontal displacement at the beam-column joint. The results of the percent errors in the calculated frequencies, obtained with these four coordinate selection schemes, are presented in Table 4. For each iteration step, there are four rows showing results corresponding to the four selection schemes described earlier. That is, the first row shows the results for the first selection scheme (that is, Shah and Raymund's method), the second row shows the results for the second selection scheme (that is, Matta's method), and so forth. It is clear from the results in this table that Shah and Raymund's scheme of coordinate selection performs the best, followed by Matta's scheme. The fourth case, where no par-

ticular scheme was used, gives completely erroneous frequencies in the first iteration step, except for the lowest frequency. Although the iterative procedure is able to significantly correct these values later, two of the seven higher frequencies have errors larger than 2% even after six iterations.

The proposed dynamic condensation procedure has also been tested on a 10-story structure shown in Fig. 2. The building is modeled as a two-dimensional frame. The structure has a total of 174 degrees of freedom. All of the beams are steel rolled sections of size W21 × 55. The column sizes in different stories are different and are as indicated in Fig. 2. At each floor level, there is an additional mass of 720 slugs. The first 10 natural frequencies obtained from the solution of the full eigenvalue problem are shown in Table 5. These values are considered as the correct values for comparison purposes.

In the solution by the proposed approach, the eigenvalue problem was first condensed to a size of 10 × 10 from 174 × 174. To define the condensation matrix, the 10 kept degrees of freedom were selected in three different ways according to 1) the lowest values of the ratio $r_i = K_{ii}/M_{ii}$, 2) Matta's scheme, and 3) Shah and Raymund's scheme. The degrees of freedom selected according to methods 1 and 2 were the same as 170, 164, 155, 149, 140, 134, 125, 119, 110, and 104. They correspond to the transverse displacements at the midspan of the beams in the two upper floors (see Fig. 2). It is apparent that this particular selection may not lead to reasonably accurate predictions of the lower frequencies. Indeed, the frequencies calculated in the first iteration step with this selection of coordinates are completely off the mark, as seen from the results in Table 6, which now only shows the percent error in the calculated values. After the first iteration, however, the errors in the lower frequencies decrease noticeably, although the higher frequencies still show large errors. After nine iterations, the three higher frequencies have errors between 5 and 13%. Evidently, the selection of coordinates based on the ratios K_{ii}/M_{ii} and Matta's scheme do not provide very satisfactory results in this case. Shah and Raymund's scheme, on the other hand, provided better results, as is seen from the results given in Table 7. The kept degrees of freedom in this case were 172, 157, 130, 127, 34, 112, 85, 76, 49, and 170. They are also indicated in Fig. 2. A radical improvement in the frequencies calculated at the first iteration step is observed. It is also noted that after the five iteration steps the first seven frequencies have errors less than 0.0005%, although the tenth frequency still has an error of more than 10%.

It is emphasized that the results in Tables 6 and 7 were obtained by solving a much smaller eigenvalue problem of size 10 × 10, which is only 6% of the full-size problem. If one is interested in accurate values of only a first few eigenproperties, this approach will prove to be remarkably efficient. Furthermore, the accuracy of the solution can be increased, if desired, by performing more numbers of iterations. However, if one wants to decrease the number of iterations for a desired accuracy or increase the accuracy for a given number of iterations, the size of the condensed eigenvalue problem must be increased. To show this, the errors in the frequencies 1, 3, 5, 7, and 9 are shown in Table 8 for increasing sizes of the condensed eigenvalue problem. These results clearly show that, the larger the size of the condensed eigenvalue problem, the smaller the error in a frequency for a given number of iterations. Or, to reduce the number of iterations required to obtain a certain number of lower frequencies with a desired accuracy, the number of the kept degrees of freedom or the size of the condensed eigenvalue problem must be increased.

Concluding Remarks

An improved dynamic condensation method is presented to calculate accurate values of the natural frequencies and modes of vibration of large structural systems. The basic idea of the proposed method is to iteratively update an estimate of the condensation matrix that relates the kept and reduced degrees of freedom. To define the initial condensation matrix required to start the iteration process, it is desirable to make a careful choice about the kept degrees of freedom. Although the eigenproperties calculated with the iterative process will converge to the exact values independently of a choice of the kept degrees of freedom, a better choice leads to a faster convergence to the results and, thus, a smaller number of iterations. Here, several ways of selecting these kept degrees of freedom are examined and compared. The method proposed by Shah and Raymund,[20] although slightly more computationally expensive than other techniques, was found to provide the best selection of the kept degrees of freedom. For calculating a given number of eigenproperties, a larger sized condensed problem requires a smaller number of iterations to provide a desired level of accuracy. Thus, the process involves a tradeoff. Although no optimization study was performed here, it is felt that, to keep the iteration steps at a probable minimum and yet obtain accurate results, the size of the condensed eigenvalue problem should be about 40% larger than the number of the frequencies of interest.

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